TWO EXACT SOLUTIONS OF THE EQUATIONS OF HYDRODYNAMICS OF THE TRIPLE-WAVE TYPE

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In the papers [1 and 2] are given exact solutions of the problem of plane unsteady flow of isothermal gas, arising in the motion of two perpendicular pistons, and of the problem of discharge into a vacuum along on oblique wall of polytropic gas for $1 < \gamma < 3$ (where γ is the adiabatic index). The solutions consist of double and simple waves [1].

In the present paper we consider the analogous problems for the three-dimensional case. The exact solution is given of the problem of the flow of an isothermal gas, bounded by three moving mutually orthogonal planes, and also of the problem of discharge of a plytropic gas into a vacuum along a certain two-sided angle, for $1 < \gamma < 2$. In these solutions there occurs a sequence of contiguous waves of rank 0, 1, 2 and 3. The generalization to the three-dimensional case is not trivial, since triple waves are described by an over-determined nonlinear system of partial differential equations with a complicated structure, the study of the compatibility of which is very difficult.

1. In the hodograph space u_1, u_2, u_3 (u_i are the components of the velocity vector) the system of equations, describing triple waves under the assumption of potential and isentropic flow, has the form [3]

$$\sum_{ik} A_{ik}^{\circ} L_{ik}^{j} = 0 \qquad \left(\begin{matrix} j = 0, 1, 2 \\ i, k = 1, 2, 3 \end{matrix}\right), \qquad A_{ik}^{\circ} = \delta_{ik}\theta - \varkappa^2\theta_i\theta_k \qquad \left(\varkappa = \frac{1}{\gamma - 1}\right) \tag{1.1}$$

$$L_{ik}^{\circ} = (-1)^{i+k} \begin{vmatrix} \varkappa\Pi_{mp} + \delta_{mp} & \varkappa\Pi_{np} + \delta_{np} \\ \varkappa\Pi_{mq} + \delta_{mq} & \varkappa\Pi_{nq} + \delta_{nq} \end{vmatrix} \tag{1.2}$$

$$L_{ik}^{1} = (-1)^{i+k} \left\{ \begin{vmatrix} \varkappa\Pi_{mp} + \delta_{mp} & \varkappa\Pi_{np} + \delta_{np} \\ \varkappa\theta_{mq} + \delta_{mq} & \varkappa\theta_{nq} + \delta_{nq} \end{vmatrix} + \begin{vmatrix} \varkappa\theta_{mq} + \delta_{mp} & \varkappa\theta_{np} + \delta_{np} \\ \varkappa\Pi_{mq} + \delta_{mq} & \varkappa\Pi_{nq} + \delta_{nq} \end{vmatrix} \right\} \tag{1.3}$$

$$L_{ik}^{2} = (-1)^{i+k} \begin{vmatrix} \varkappa\theta_{mp} + \delta_{mp} & \varkappa\theta_{np} + \delta_{np} \\ \varkappa\theta_{mq} + \delta_{mq} & \varkappa\theta_{nq} + \delta_{nq} \end{vmatrix} \tag{1.4}$$

$$\Pi = \Pi(u_1, u_2, u_3), \qquad \theta = \theta(u_1, u_2, u_3), \qquad \Pi_{ik} = \frac{\partial^2\Pi}{\partial u_i \partial u_k}, \qquad \theta_{ik} = \frac{\partial^2\theta}{\partial u_i \partial u_k} \tag{1.5}$$

$$(m, n \neq k;\ m < n;\ p, q \neq i;\ p < q)$$

Here γ is the adiabatic index, θ is the square of the velocity of sound, δ_{ik} is the Kronecker symbol. After finding the functions Π and θ from the system of three equations (1.1), the motion in physical space x_1, x_2, x_3, t is determined by the relations

$$x_i = \varkappa\Pi_i + u_i + t\,(\varkappa\theta_i + u_i) \quad (i = 1, 2, 3) \qquad \left(\Pi_i = \frac{\partial \Pi}{\partial u_i}, \quad \theta_i = \frac{\partial \theta}{\partial u_i}\right) \quad (1.6)$$

The equations for isothermal gas can be obtained by setting formally $\varkappa = 1$ in (1.2) to (1.6), introducing instead of θ the function $q = \ln \rho$ (where ρ is the density of the gas) and instead of A°_{ik} the expressions

$$A_{ik}^1 = \delta_{ik} - q_i q_k, \quad q_i = \partial q / \partial u_i \quad (1.7)$$

Let us consider the equations for triple waves in isothermal gas with the equation of state $p = \rho$ (where p is the pressure, the isothermal velocity of sound being set equal to unity). The last equation of the system (1.1) for $j = 2$ is satisfied by a function of the form

$$q = u_1 + u_2 + u_3 + C \qquad (C = \text{const}) \quad (1.8)$$

The two remaining equations of the system (1.1) for q from (1.8) take the form

$$\begin{vmatrix} \Pi_{21} & \Pi_{31} \\ \Pi_{23} & \Pi_{33}+1 \end{vmatrix} - \begin{vmatrix} \Pi_{21} & \Pi_{31} \\ \Pi_{22}+1 & \Pi_{32} \end{vmatrix} + \begin{vmatrix} \Pi_{11}+1 & \Pi_{31} \\ \Pi_{12} & \Pi_{32} \end{vmatrix} = 0 \quad \text{for } j = 0 \quad (1.9)$$

$$\Pi_{21} + \Pi_{31} + \Pi_{32} = 0 \qquad \text{for } j = 1 \quad (1.10)$$

Making use of (1.10), we can express Equation (1.9) as follows:

$$\begin{vmatrix} \Pi_{21} & \Pi_{31} \\ \Pi_{23} & \Pi_{33} \end{vmatrix} - \begin{vmatrix} \Pi_{21} & \Pi_{31} \\ \Pi_{22} & \Pi_{32} \end{vmatrix} - \begin{vmatrix} \Pi_{11} & \Pi_{31} \\ \Pi_{12} & \Pi_{32} \end{vmatrix} = 0 \quad (1.11)$$

The system of equations (1.10) and (1.11) for the function Π has a solution of the form

$$\Pi = f_1(u_1) + f_2(u_2) + f_3(u_3) \quad (1.12)$$

Here the functions f_i are arbitrary.

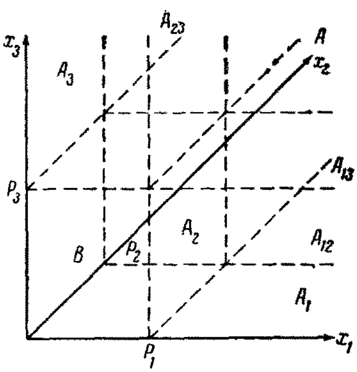

Fig. 1

Accordingly, for isothermal gas we find a solution of the system of equations for triple waves, depending on three arbitrary functions of one independent variable, with the function q defined by (1.8).

Let us make use if the solution so obtained for finding the flow of gas bounded by three mutually orthogonal moving planes.

Let the isothermal gas at the initial instant of time $t = 0$ be included at rest inside a three-sided corner, bounded by the planes $x_i = 0$ ($i = 1, 2, 3$). At the instant of time $t = 0$ the planes start to move according to the law

$$x_i = F_i(t) \quad (1.13)$$

where the $F_i(t)$ are such that up to a certain moment of time T there do not appear in the motion any strong discontinuities. The case with discontinuities can be considered quite analogously. This is done explicitly for two-dimensional case in [1].

Then for $0 < t < T$ in the x_1, x_2, x_3 space we shall have the following pattern of flow (Fig.1). The planes P_i are planes of weak discontinuities, and the equations of their motions can be written in the form $x_i = t$ (the velocity of sound being taken equal to unity).

As a whole, the motion is three-dimensional. In the region A, bounded by the planes P_i, it is at rest. In the regions A_{ik} $(i \neq k)$, bounded by the planes $x_j = 0$ $(j \neq i,\ j \neq k)$ and P_l $(l = 1, 2, 3)$, we shall have one-dimensional motion, in the form of a travelling Riemann wave.

In the region A_i $(i = 1, 2, 3)$, bounded by the planes $x_k = 0$, $x_j = 0$ $(k, j \neq i)$ and P_i, we shall have two-dimensional motion of double-wave type, similar to that considered in [1]. Finally, in region B, bounded by the planes $x_i = 0$ and P_i, there will be motion in the form of a wave of rank three.

From the condition for continuity of the solution in the regions A_i with the travelling Rieman waves at the planes P_j, P_k $(j \neq i,\ k \neq i$ for $A_i)$ there follow the relations

$$q = u_k + q_0 \quad \text{for} \quad u_i = 0,\ u_j = 0; \qquad q = u_j + q_0 \quad \text{for} \quad u_i = 0, \quad u_k = 0 \quad (1.14)$$

From similar considerations for the region B at the planes P_i, the following boundary conditions are established (with the help of the already known solution for double waves):

$$q = u_k + u_j + q_0 \quad (i = 1, 2, 3;\ k, j \neq i) \quad \text{for} \quad u_i = 0 \qquad (1.15)$$

Accordingly, we can satisfy conditions (1.14) and (1.15), if in (1.8) we set $C = q_0$. The question of the uniqueness of this solution remains open. Finally, let us introduce the following summary for the solution of the problem formulated:

$$u_i = 0 \quad \text{in region } A \qquad (1.16)$$

$$u_j = u_k = 0 \quad (k, j \neq i), \quad u_i = u_i(x_i, t), \quad q = u_i + q_0 \text{ in region } A_i$$

$$u_i = 0, \quad u_j = u_j(x_j, t), \quad u_k = u_k(x_k, t) \quad (k, j \neq i), \quad q = u_j + u_k + q_0 \text{ in region } A_{jk}$$

$$u_i = u_i(x_i, t), \quad q = u_1 + u_2 + u_3 + q_0 \text{ in region } B$$

Here $i = 1, 2, 3$, whilst the functions $u_i(x_i, t)$ are found from Equations

$$x_i = f_i'(u_i) + u_i + t(1 + u_i) \qquad (1.17)$$

arising from (1.6), whilst the f_i, in accordance with (1.13), are determined from Equations

$$F_i'(t)\,t + F_i'(t) + f_i'[F_i'(t)] = F_i(t) - t \qquad (i = 1, 2, 3) \qquad (1.18)$$

Similar formulas also hold for the solutions with different types of discontinuity.

2. Let us consoder a gas for which the equation of state is

$$p = a^2\rho^\gamma \quad (a^2 = \text{const}).$$

We shall seek a solution of the system (1.1) in the form

$$\Pi = \theta = \theta(r) = \theta(a_0 + a_1u_1 + a_2u_2 + a_3u_3) \quad (a_0, a_1, a_2, a_3 = \text{const}) \quad (2.1)$$

In this case all three equations of the system (1.1) are the same and have the form

$$3\theta + 2\varkappa A\theta\theta'' - \varkappa^2 A\theta'^2 = 0 \qquad (A = a_1^2 + a_2^2 + a_3^2) \qquad (2.2)$$

Integrating (2.2), we obtain

$$r = \int \left(C_1\theta^\varkappa + \frac{3\theta}{\varkappa A(\varkappa - 1)}\right)^{-1/2} d\theta + C_2 \qquad (2.3)$$

Here C_1 and C_2 are arbitrary constants. With $C_1 \neq 0$, there is under the integral sign of (2.3) a polynomial expression, and according to well known criteria of integral (2.3) is expressed by elementary functions only for $\gamma = 2 \pm (2k + 1)^{-1}$, where k is an integer. With the help of (2.3) and (1.6) we can obtain a family of exact solutions of the equations of hydrodynamics of the triple-wave type, depending on the constants C_1 and a_i $(i = 0, 1, 2, 3)$.

The solutions under consideration are three-dimensional self-similar flows with the independent delf-similar variables $\xi_i = (1 + t)^{-1}x_i$ $(i = 1, 2, 3)$.

When $C_1 = 0$ we shall have for the sound velocity c from (2.3)

$$c = \alpha_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \qquad (\theta = c^2) \tag{2.4}$$

Moreover,

$$\alpha_1^2 + \alpha_2^2 + \alpha_2^3 = \frac{3}{4\varkappa\ (\varkappa - 1)} \tag{2.5}$$

We shall show that with the help of (2.4) we can solve the problem of discharge of gas into a vacuum from a certain infinite three-sided corner, one of the sides of which is instantaneously removed, whilst the discharge occurs along the continuation of the planes of the other two sides. Suppose that in the gas at rest $c = 1$. In (2.4) let us set

$$\alpha_0 = 1, \quad \alpha_1 = h, \quad \alpha_2 = h \cot \alpha, \quad \alpha_3 = h \cot \beta$$
$$h = \frac{\gamma - 1}{2}, \qquad \cot \alpha = \left(\frac{1+\gamma}{3-\gamma}\right)^{1/2}, \qquad \cot \beta = \left(\frac{1+\gamma}{(3-\gamma)(2-\gamma)}\right)^{1/2} \tag{2.6}$$

Here a_1 and a_2 are defined so that when $u_3 = 0$ a double wave results, solving the problem of plane discharge into a vacuum along an oblique wall (see [2]), whilst a_3 is determined from the condition (2.5). From it follows that the discussion is valid only for $\gamma < 2$. The flow in the space of the self-similar variables ξ_i is found from the linear system

$$\begin{aligned} 1 + hu_1 + h \cot \alpha u_2 + h \cot \beta\, u_3 + u_1 &= \xi_1 \\ \cot \alpha\, (1 + hu_1 + h \cot \alpha u_2 + h \cot \beta\, u_3) + u_2 &= \xi_2 \\ \cot \beta\, (1 + hu_1 + h \cot \alpha\, u_2 + h \cot \beta\, u_3) + u_3 &= \xi_3 \end{aligned} \tag{2.7}$$

the determinant of which depends upon γ and does not vanish identically. In Figs. 2 and 3 are given the regions of flow in the hodograph space and in the ξ_1, ξ_2, ξ_3 space, when the gas at the initial instant of time occupied the three-sided corner bounded by the planes

$$x_1 = 0, \qquad x_2 = x_1 \cot \alpha, \qquad x_3 \cot \alpha = x_2 \cot \beta \tag{2.8}$$

for $x_i \geqslant 0$, and then the side $x_1 = 0$ is removed.

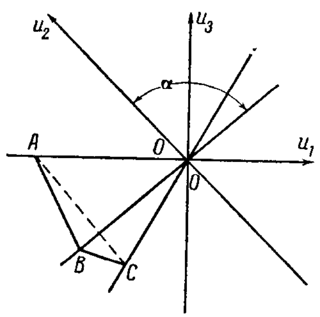

Fig. 2

In the hodograph space the regions of flow correspond to the tetrahedron $ABCO$, bounded by the planes

$$(S_1)\ u_3 = 0, \quad (S_2)\ u_2 = u_1 \cot \alpha, \quad (S_3)\ u_3 \cot \alpha = u_2 \cot \beta \tag{2.9}$$
$$(S_4)\ 1 + hu_1 + h \cot \alpha\, u_2 + h \cot \beta\, u_3 = 0$$

In ξ_1, ξ_2, ξ_3 space the tetrahedron $A'B'C'O'$, bounded by the planes

$$(S_1')\quad \cot \beta + h \cot \beta\, \xi_1 + h \cot \alpha \cot \beta\, \xi_2 - (1 + h + h \cot^2 \alpha)\, \xi_3 = 0$$
$$(S_2')\ \xi_2 = \xi_1 \cot \alpha, \qquad (S_3')\ \xi_3 \cot \alpha = \xi_2 \cot \beta \tag{2.10}$$
$$(S_4')\ 1 + h\xi_1 + h \cot \alpha \xi_2 + h \cot \beta\, \xi_3 = 0$$

corresponds to the triple wave. The region lying above the plane (S_1') and bounded by the three planes, passing through each pair of the straight lines l_i $(i = 1, 2, 3)$, parallel to the ξ_3-axis, corresponds to the double wave; in the hodograph space this region corresponds to the plane (S_1).

Finally, the region bounded by the plane (S_3'), the plane passing through the straight lines l_1 and l_3, the planes $\gamma_1 l_1$, $\gamma_3 l_3$, passing through the lines l_1 and l_3 orthogonally to the ξ_1-axis, is the region of the travelling Riemann wave with $u_2 = u_3 = 0$. In the hodograph space it transforms into the line AO . We notice that the angle between the planes (S_2') and (S_3') , along which the discharge occurs, does not depend upon γ and is

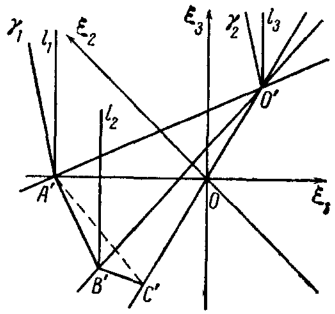

Fig. 3

equal to $\pi/3$. The front of the discharge in the vacuum ($\rho = 0$) is formed by the three planes (S_4'), $l_1\gamma_1$ and $l_1 l_2$, intersecting at the point A' . Moreover, (S_4') is orthogonal to the planes (S_2') and (S_3') , the plane $l_1\gamma_1$ is orthogonal to (S_3') and the plane $l_1 l_2$ is orthogonal to (S_2') . The plane $\gamma_2 l_3$ corresponds to the front of the weak discontinuity propagating into the stationary gas.

BIBLIOGRAPHY

1. Pogodin, Iu.Ia., Suchkov, V.A. and Ianenko, N.N., O begushchikh volnakh uravnenii gazovoi dinamiki (On travelling waves of the equations of gasdynamics). Dokl.Akad.Nauk SSSR, Vol.119, № 3, 1958.
2. Suchkov, V.A., Istechenie v vakuum na kosoi stenke (Flow into a vacuum along an oblique wall). *PMM* Vol.27, № 4, 1963.
3. Sidorov, A.F., O nestatsionarnykh potentsial'nykh dvizheniiakh politropnogo gaza s vyrozhdennym godografom (On nonstationary potential motions of a polytropic gas with a degenerate hodograph). *PMM* Vol.23, № 5,1959.

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